Distributivity in Set Theory — Part I —

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Abstract We will introduce the distributive phenomena which appear in several arguments on the modern set theory, i.e., the axiomatic set theory. In Part 1, several notations and definitions are given, and elementary results are presented.

§1. Preface

In the usual set theory, say $ZFC$ set theory, the distributive law:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

is easily proved.

The notion is generalized to one for infinitely many sets. For example,

$$A \cap \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} (A \cap B_i)$$

The distributive law is also fundamental property for Boolean algebras. Why does the distributive law appear in so many set theoretical contexts? A reason is given in my doctoral thesis [7]:

‘Thus, when we construct and develop a powerful set theory based on Zermelo-Fraenkel set theory, it happens quite often to find out one condition, say $h(\alpha)$, from each set of conditions, say $A_\alpha$, whose disjunction is consistent (i.e., $\bigvee_{\alpha<\kappa} A_\alpha = 1$ in Boolean terms) and arrange them into one consistent condition (i.e., $\bigwedge_{\alpha<\kappa} h(\alpha) > 0$ in Boolean terms). Just as we can see in many concrete proofs, distributivity can be regarded as a logical principle which can make us put the above process into practice. We shall get a view of how distributive phenomenon appears in the modern set theory and express it uniformly.’

So, in this note, we will introduce the distributive phenomena which appear in several arguments on the modern set theory, i.e., the axiomatic set theory.

Remarks. In this chapter, we work in the axiomatic set theory of Zermelo-Fraenkel, abbreviated by $ZF$. If we need other axioms, say the Axiom of Choice abbreviated by $AC$, we indicate it explicitly.

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§2. Notations and Terminologies

We use standard set theoretic notation (see Jech[3]). For example, $|X|$ denotes the cardinality of the set $X$ (if such a cardinal exists), whereas $|A| \leq |B|$ and $|A| = |B|$ indicate that there exists an injection of $A$ into $B$ and that there exists a bijection of $A$ onto $B$, respectively. $|A| < |B|$ means that $|A| \leq |B|$ but not $|A| = |B|$.

For any sets $A$ and $X$, we set:

$\mathcal{P}_{\leq |A|}(X) = [X]_{\leq |A|} = \{ Y \subseteq X : |Y| \leq |A| \}$,

$\mathcal{P}_{\leq |A|}(X) = [X]_{\leq |A|} = \{ Y \subseteq X : |Y| < |A| \}$,

and $\mathcal{P}_{|A|}(X) = [X]_{|A|} = \{ y \subseteq X : |y| = |A| \}$.

Clearly, we get $\mathcal{P}(X) = \mathcal{P}_{\leq |X|}(X)$.

Small Greek letters $\alpha, \beta, \gamma, \ldots$ denote ordinals, and the Greek letters $\kappa, \lambda, \mu, \ldots$ are reserved for denoting cardinals. As usual, we write $\kappa$ instead of $|\kappa|$, for cardinal $\kappa$.

By $\aleph_\alpha$, we indicate the $\alpha$-th infinite cardinal.

For each function $f$, we set:

$\Sigma f = \{ (x, y) : x \in \text{dom}(f) \text{ and } y \in f(x) \}$

and $\Pi f = \{ g \in \text{dom}(f) \cup \text{ran}(f) : \forall x \in \text{dom}(f) \ g(x) \in f(x) \}$,

where $(x, y)$ is the ordered pair of $x$ and $y$, and $^A B$ is the family of all functions of $A$ into $B$.

Let $C_{x, y}$ or $(x, y)$ denote the constant function on $x$ whose range is the singleton $\{y\}$.

Then, the following are established:

$\kappa + \lambda = |\Sigma \{ \langle 0, \kappa \rangle, \langle 1, \lambda \rangle \}|$,

$\kappa \times \lambda = |\Pi \{ \langle 0, \kappa \rangle, \langle 1, \lambda \rangle \}|$

and $\kappa^\lambda = |^A \lambda| = |\Pi (\lambda; \kappa)|$,

where $+$ and $\times$ denote the cardinal sum and product, respectively, whereas $+$ and $\cdot$ indicate the ordinal sum and product, respectively.

We assume a familiarity with the definitions and basic properties of Boolean algebras. An ideal $I$ in a Boolean algebra $B$ is a subset of $B$ such that:

i) $0 \in I$ and $1 \notin I$,

ii) if $a, b \in I$ then $a \lor b \in I$,

iii) if $a \in I$ and $b \in B$ then $a \land b \in I$.

Then the dual set $I^*$ consisting of all complements $a^*$ of $a$ in $I$ forms a filter in $B$.

Now, we present a result on the completions of partially ordered sets.
Lemma 2.1. ([2]) Let \( \langle P, \langle \rangle \rangle \) be a partially ordered set. Then there is a complete Boolean algebra \( \langle B, \langle \rangle \rangle \) and a function \( i \) of \( P \) into \( B^+ = B - \{0\} \) such that:

1. \( \text{ran}(i) \) is dense in \( B^+ \), i.e. \( \forall a \in B^+ \exists p \in P \ (i(p) \leq a) \).
2. \( \forall p, q \in P \ (p \leq q \Rightarrow i(p) \leq i(q)) \).
3. \( \forall p, q \in P \ (\forall r \in P \ (r \not\leq p \text{ or } r \not\leq q) \Rightarrow i(p) \land i(q) = 0) \).

Proof. At first, we define an equivalence relation \( \sim \) by:

\[ p \sim q \text{ if and only if for any } r \in P, \text{ the condition that there is } s \in P \text{ such that } s \leq p \text{ and } s \leq r, \text{ is equivalent to that there is } s' \in P \text{ such that } s' \leq q \text{ and } s' \leq r. \]

Then it is easily to check that \( \sim \) is an equivalence relation.

Let \( j(p) \) be the equivalence class including \( p \) for each \( p \in P \).

Then, the set \( \{ j(p) \mid p \in P \} \) forms a refined partially ordered set \( \langle Q, \leq \rangle \) and \( j \) is an order preserving function from \( P \) onto \( Q \), where the refinement of \( Q \) means that \( Q \) satisfies that for any members \( p \) and \( q \) of \( Q \), if \( q \not\leq p \), then there is \( p' \) in \( Q \) such that \( p' \leq q \) and there is no \( s \in Q \) with \( s \leq p \) and \( s \leq p' \).

Next, putting \( k(p) = \{ q \in Q \mid q \leq p \} \), we can get an embedding \( k \) from \( Q \) into a complete Boolean algebra \( B \) consisting of all regular open sets of \( Q \) with the partial ordering the inclusion \( \subseteq \). In the refined Boolean algebra \( Q \), it holds that for all \( p \in Q \) the set \( k(p) \) is regular open, that is, in the order topology which has the set of all \( k(q) \) \( (q \in Q) \) as its base, \( k(p) \) is equal to the interior of the closure of itself.

Thus, it can be noticed that the function \( i = k \circ j \) is an isomorphism from \( P \) onto a dense subset of the complete Boolean algebra \( B \). Then, it is easy to check that the conditions (1), (2) and (3) hold.

In the above, \( B \) is called the completion of \( P \).

Hereafter, we assume that \( B \) is some fixed Boolean algebra.

Definition 2.2. A subset \( X \) of \( B \) is said to be \( \lambda \)-complete if the sum of every subset of \( X \) of cardinality less than \( \lambda \) exists in \( X \), and to be complete if the sum of every subset of \( X \) exists in \( X \).

Definition 2.3. Let \( I \) be any fixed ideal in \( B \). A subset \( X \) of \( B \) is said to be \( I \)-disjoint if \( X \subseteq I^+ = B - I \) and \( x \wedge y \in I \) for any distinct members \( x \) and \( y \) of \( X \). If \( X \) is \( I_0 \)-disjoint, it is simply said to be disjoint, where \( I_0 \) is the trivial ideal \( \{0\} \) in \( B \). Moreover, if the sum of a disjoint subset \( X \) of \( B \) is \( b \), \( X \) is called a partition of \( b \).

We usually write \([a]_I \) (or simply \([a] \)) to denote the coset co-
responding to $a$ in the quotient algebra $B/I$. Then if the family
\{\{a\}_I : a \in X\} is a partition of $[b]_I$ in $B/I$, $X$ is said to be an
$I$-partition of $b$. $B$ is $\lambda$-saturated if there is no disjoint subsets of
cardinality $\lambda$, and $I$ is $\lambda$-saturated if $B/I$ is $\lambda$-saturated.

Now, we introduce power set algebras as interesting examples of
Boolean algebras and as useful examples of Boolean algebras.

**Definition 2.4.** For any set $X$, a field of sets $\mathcal{F}$ of $X$ means a
non-empty set of subsets of $X$ satisfying:
1) if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$,
2) if $A \in \mathcal{F}$, then $X - A \in \mathcal{F}$.
3) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
$\mathcal{F}$ is said to be ‘$\lambda$-complete’ provided
4) if $G \subseteq \mathcal{F}$ and $|G| < \lambda$, then $\bigcup G \in \mathcal{F}$.

For any topological space $X$, the set of all clopen sets is a field of
sets, where clopen means closed and open.

**Definition 2.5.** For a field of sets $\mathcal{F}$, an ideal $I$ on $\mathcal{F}$ is a subset of
$\mathcal{F}$ such that:
1) $\emptyset \in I$ and $X \notin I$,
2) if $A, B \in I$, then $A \cup B \in I$.
3) if $A \in I$ and $B \in \mathcal{F}$, then $A \cap B \in I$.
Moreover, under the assumption that $\mathcal{F}$ is $\lambda$-complete,
$I$ is said to be $\lambda$-complete if it satisfies that
4) if $J \subseteq I$ and $|J| < \lambda$, then $\bigcup J \in I$.

Next, consider ideals in the power set algebras $\mathcal{P}(S)$. Let $I$ be an
ideal on a set $S$ which is partially ordered by $\preceq$.
At first, we introduce some combinatorial notions.

**Definition 2.6.** Let $A$ be any subset of $S$.
1) $A$ is $\prec$-bounded if $\exists a \in S \ \forall x \in A \ \ x \prec a$.
2) $A$ is $\prec$-unbounded if $\forall x \in S \ \exists a \in A \ \ x \prec a$.
3) $A$ is $\prec$-closed if whenever $\langle a_\xi : \xi < \mu \rangle$ is any $\prec$-increasing sequence
   in $A$ with $b = \sup_{\xi < \mu} a_\xi$, $b$ is also in $A$.
4) $A$ is $\prec$-club if $A$ is both $\prec$-unbounded and $\prec$-closed.
5) $A$ is $\prec$-stationary if for every $\prec$-club $C$ in $S$, $A \cap C \neq \emptyset$.
6) $A$ is $\prec$-thin if for some $\prec$-club $C$ in $S$, $A \cap C = \emptyset$, i.e. $A$ is not
   $\prec$-stationary.
In the above, \( b = \sup_{\xi \prec \mu} a_\xi \) indicates the least \( \prec \)-upper bound of \( \{ a_\xi \mid \xi \prec \mu \} \), i.e., \( b \) satisfies the following conditions:

1) for any \( \xi \prec \mu \), \( a_\xi \prec b \),
2) if \( c \in S \) satisfies that for any \( \xi \prec \mu \), \( a_\xi \prec c \), then \( b = c \) or \( b \prec c \).

Next, let \( I \) be an ideal on \( S \).

**Definition 2.7.** Let \( f \) be a function on a subset \( A \) of \( S \).

1) \( f \) is \( \prec \)-small if for every \( x \in \text{ran}(f) \), \( f^{-1}(\{x\}) \in I \).
2) \( f \) is \( \prec \)-injective on \( D \subseteq A \) if whenever \( a, b \in D \) and either \( a \prec b \) or \( b \prec a \), \( f(a) \neq f(b) \).
3) \( f \) is \( \prec \)-regressive if \( \text{ran}(f) \subseteq S \) and for each \( a \in A \) if \( \text{pr}_{\prec}(a) \neq \emptyset \) then \( f(a) \prec a \) and otherwise \( f(a) = a \), where \( \text{pr}_{\prec}(a) \) is the set of all predecessors of \( a \) in \( S \), i.e., \( \text{pr}_{\prec}(a) = \{ b \in S : b \prec a \} \).

Let \( h \) be any function of \( S \) into \( \mathcal{P}(S) \).

4) \( X \) is the \( \prec \)-diagonal intersection of \( h \), \( X = \triangleleft h \) (or simply \( X = \triangleleft h \)), if \( X = \{ x \in S : \forall y < x \ x \in h(y) \} \).
5) \( X \) is the \( \prec \)-diagonal union of \( h \), \( X = \triangleright h \) (or simply \( X = \triangleright h \) if \( X = \{ x \in S : \exists y < x \ x \in h(y) \} \).

**Definition 2.8.** (1) \( I \) is said to be non-trivial if for every \( a \in S \), \( \{ a \} \in I \).
(2) \( I \) is said to be \( \prec \)-fine if whenever \( A \) is not \( \prec \)-unbounded, \( A \in I \).
(3) \( I \) is said to be \( \prec \)-normal if for any function \( h \) of \( S \) into \( I \), \( \triangleright h \in I \).

Thus, each \( \prec \)-fine ideal is non-trivial.

**Lemma 2.9 (AC).** Assume that \( I \) is a \( \prec \)-fine ideal and the set \( A = \{ a \in S : \text{pr}_{\prec}(a) = \emptyset \} \) has \( I \)-measure zero, i.e., \( A \in I \). Then \( I \) is \( \prec \)-normal if and only if every \( \prec \)-regressive function on a set \( D \) in \( I^+ \) is not \( I \)-small.

**Proof.** ‘If part’: Assume that \( I \) is a \( \prec \)-fine ideal and let \( h \) be any function of \( S \) into \( I \), and for each \( x \in \triangleright h \), let \( y_x \) be any element of \( S \) such that \( x \in h(y_x) \) and \( y_x \prec x \).

Such selection is made possible by Axiom of Choice.

Then clearly the function \( f \) which maps \( x \) to \( y_x \) is \( \prec \)-regressive on \( \triangleright h \). If \( \triangleright h \) is in \( I^+ \), by the assumption, \( f \) is not \( I \)-small, i.e.,
there is $y \in \text{ran}(f)$ with $f^{-1}(\{y\}) \subseteq I^+$. 

This implies that $f^{-1}(\{y\}) \subseteq h(y)$ and as $h(y) \in I$, $f^{-1}(\{y\})$ is also in $I$. This is a contradiction. Hence $\nabla^\prec h \notin I$. Thus, $I$ is $\prec$-normal.

‘Only if part’: Assume that $I$ is $\prec$-normal and $f$ is a $\prec$-regressive function on a set $D$ in $I^+$. And moreover we assume that $f^{-1}(\{y\})$ is in $I$ for all $y \in \text{ran}(f)$.

Let $h$ be the function of $\text{ran}(f)$ into $I$ defined by $h(y) = f^{-1}(\{y\})$ for each $y \in \text{ran}(f)$. Then, for each $x \in D$, $f(x) \prec x$ and $x \in f^{-1}(\{f(x)\}) = h(f(x))$, and so $x \in \nabla^\prec h$.

Hence, as $D \subseteq \nabla^\prec h$ and $D \in I^+$, $\nabla^\prec h$ is in $I^+$. This contradicts that $I$ is $\prec$-normal. Therefore, for any $\prec$-regressive function $f$ on a set $D$ in $I^+$, $f$ is not $I$-small.

**Lemma 2.10 (AC).** Let $BD_S^\prec = \{X \subseteq S : X$ is not $\prec$-unbounded $\}$ and $NS_S^\prec = \{X \subseteq S : X$ is $\prec$-thin $\}$. Assume that every $\prec$-increasing sequence in $S$ of length $\omega$ has its $\prec$-supremum (in $S$). Then:

1. If each subset $A$ of $S$ of cardinality less than $\kappa$ is $\prec$-bounded, then $BD_S^\prec$ is a $\prec$-fine $\kappa$-complete ideal on $S$.

2. If each $\prec$-increasing sequence in $S$ of length less than $\kappa$ is $\prec$-bounded, then $NS_S^\prec$ is a $\prec$-fine $\kappa$-complete ideal on $S$.

**Proof.** (1) Let $\{X_\xi \mid \xi < \eta \}$ be a subfamily of $BD_S^\prec$ with $\eta < \kappa$. As $X_\xi$ is not $\prec$-unbounded, for each $\xi < \eta$, there is $a_\xi \in S$ with $a_\xi \not\prec x$ for all $x \in X_\xi$. Since the set $\{a_\xi \in S \mid \xi < \eta \}$ has cardinality less than $\kappa$, it is $\prec$-bounded, i.e., there is $a \in S$ with $a_\xi \not\prec a$ for all $\xi < \eta$.

Let $x$ be any in $\bigcup_{\xi < \eta} X_\xi$, then for some $\xi < \eta$, $a_\xi \not\prec x$, and so $a \not\prec x$. Otherwise it holds that $a_\xi \prec a < x$, which contradicts $a_\xi \not\prec x$. Hence $\bigcup_{\xi < \eta} X_\xi$ is not $\prec$-unbounded, and $\bigcup_{\xi < \eta} X_\xi$ is in $BD_S^\prec$. This means that $BD_S^\prec$ is $\kappa$-complete. It is trivial that it is $\prec$-fine.

(2) Let $\{X_\xi \mid \xi < \eta \}$ be a subfamily of $NS_S^\prec$ with $\eta < \kappa$. Then, for each $\xi < \eta$, there is a $\prec$-club set $C_\xi$ with $X_\xi \cap C_\xi = \emptyset$. Clearly $\bigcup_{\xi < \eta} X_\xi \cap \bigcap_{\xi < \eta} C_\xi = \emptyset$, so we have only to prove that $\bigcap_{\xi < \eta} C_\xi$ is $\prec$-club. It is easy to check that $\bigcap_{\xi < \eta} C_\xi$ is $\prec$-closed. So we concentrate on proving that $\bigcap_{\xi < \eta} C_\xi$ is $\prec$-unbounded.

Let $x$ be any in $S$. Since $C_0$ is $\prec$-unbounded, there is $a_{0,0} \in C_0$ with $x < a_{0,0}$, and as $C_1$ is $\prec$-unbounded, there is $a_{1,0} \in C_1$ with $a_{0,0} < a_{1,0}$, and so on. Thus, we can construct a $\prec$-increasing sequence $\langle a_{\xi,0} \mid \xi < \eta \rangle$ such that $x < a_{0,0}$ and $a_{\xi,0} \in C_\xi$ for each $\xi < \eta$.

Moreover, we can construct a $\prec$-increasing $\langle a_{\xi,n} \mid \xi < \eta$ and $n \in \omega \rangle$ such that $a_{\xi,m} \in C_\xi$ for any $m < \omega$, if $m < n$, $a_{\xi,m} < a_{\xi,n}$ for
any $\xi, \xi' < \eta$, and for each $m < \omega$ if $\xi < \xi' < \eta$, $a_{\xi',m} < a_{\xi,m}$.

By the assumption, there is $a_\xi \in S$ such that for any $n < \omega$, $a_\xi = \sup_{n<\omega} a_{\xi,n}$. Clearly each $a_\xi$ is in $C_\xi$. However, for all $\xi < \eta$, $a_\xi$ is equal to the same $a$, by their construction.

Thus we get that $x < a$ and $a \in \bigcap_{\xi<\eta} C_\xi$ and this implies that $\bigcap_{\xi<\eta} C_\xi$ is $\prec$-club.

\section{Definitions of Generalized Distributivity}

In this section, we define several kinds of distributivity and introduce elementary properties of those notions.

Hereafter, $B$ always indicates a fixed Boolean algebra and $I$ does an ideal in $B$.

\textbf{Definition 3.1.} ([8]) Let $f$ be any function. Then, $B$ is said to be \langle $\lambda$, $f$\rangle-distributive if $\forall b \in B$ $(\forall \in dom(f) \ 0 < b \leq \bigvee f(a) \implies \exists v \in \Pi f \forall t \in [dom(f)]^\lambda (b \land \land_{a \in t} v(a) > 0))$.

In the above, $\land$ and $\bigvee$ are the usual generalizations of $\land$ and $\bigvee$, respectively.

\textbf{Definition 3.2.} (1) $B$ is \langle $\lambda$, $A$\rangle-distributive if it is \langle $\lambda$, $f$\rangle-distributive for any function $f$ with $A = dom(f)$.

(2) $B$ is $f$-distributive if for some cardinal $\lambda$ with $|dom(f)| < \lambda$, it is \langle $\lambda$, $f$\rangle-distributive.

(3) $B$ is \langle $\lambda$\rangle-distributive if for any function $f$ with $|f(a)| \geq 1$ for all $a \in dom(f)$, it is \langle $\lambda$, $f$\rangle-distributive.

(4) $B$ is $A$-distributive if it is \langle $\lambda$, $A$\rangle-distributive for any cardinal $\lambda$.

(5) $B$ is completely (resp. disjoint) distributive if for any function $f$ with $|f(a)| \geq 1$ for all $a \in dom(f)$, it is $f$-distributive.

\textbf{Lemma 3.3.} Let $\lambda$ be any cardinal. Let $f$ be any function and assume that there are a cardinal $\kappa < \lambda$ and a surjection $h$ of $dom(f) \times \kappa$ onto $\bigcup ran(f)$ such that $\forall \langle a, \xi \rangle \in dom(f) \times \kappa$, $h(\langle a, \xi \rangle) \in f(a)$.

Then, if $B$ is \langle $\lambda$, $C_{dom(f)\times 2}$\rangle-distributive, it is \langle $\lambda$, $f$\rangle-distributive.

\textbf{Proof.} Assume that $0 < b \leq \bigvee f(a)$ for all $a \in dom(f)$. Define a function $g$ of $dom(f) \times \kappa$ into $[B]^2$ by $g(a, \xi) = \{ b \land h(\langle a, \xi \rangle), b \land (h(\langle a, \xi \rangle))^*\}$ for $\langle a, \xi \rangle \in dom(f) \times \kappa$. 

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Then clearly \( b = \bigwedge_{a, \xi \in \text{dom}(f) \times \kappa} \bigvee g(a, \xi) > 0 \). Hence, by the \( \langle \lambda, C_{\text{dom}(f) \times \kappa, 2} \rangle \)-distributivity, there exists a \( v \in \Pi g, b \land \bigwedge_{a, \xi \in t} v(a, \xi) > 0 \) for all \( t \in [\text{dom}(f) \times \kappa]^{<\lambda} \).

Let \( a \in \text{dom}(f) \) be any. If for all \( \xi < \kappa, v(a, \xi) = b \land (h(\langle a, \xi \rangle))^*, \)
then since \( \kappa < \lambda, 0 < b \land \bigwedge_{\xi < \kappa} (h(\langle a, \xi \rangle))^* = b \land (\bigvee f(a))^* = 0. \)
This is absurd. Hence for some \( \xi < \kappa v(\langle a, \xi \rangle) = b \land h(\langle a, \xi \rangle). \)
Let \( \xi_a < \kappa \) be the least such one, and define \( \bar{v}(a) = h(\langle a, \xi_a \rangle) \in \text{f}(a) \) for \( a \in \text{dom}(f). \) Then, as \( [(\langle a, \xi_a \rangle : a \in \text{dom}(f)]^{<\lambda} \subseteq [\text{dom}(f) \times \kappa]^{<\lambda}, \)
for all \( t \in [\text{dom}(f)]^{<\lambda}, b \land \bigwedge_{a \in t} \bar{v}(a) = b \land \bigwedge_{a \in t} v(\langle a, \xi_a \rangle) > 0. \) Thus \( B \) is \( \langle \lambda, f \rangle \)-distributive.

**Corollary 1.** ([1]) Assume that \( \aleph_0 \leq \kappa \leq \mu, \kappa < \lambda \) and \( B \) is \( \kappa^+ \)-complete. Then \( B \) is \( \langle \lambda, (\mu; \kappa) \rangle \)-distributive if and only if it is \( \langle \lambda, (\mu; 2) \rangle \)-distributive.

**Corollary 2.** Assume that \( \aleph_0 \leq \lambda. \) Then \( B \) is \( \langle \lambda, \lambda \rangle \)-distributive if and only if it is \( \langle \lambda, 2 \rangle \)-distributive.

**Lemma 3.4.** Let \( \kappa \) and \( \lambda \) be cardinals with \( \aleph_0 \leq \lambda \) and let \( A \) be any non-empty set with \( |A| < \lambda. \) Let \( g \) be a function such that \( 1 \leq |g(a)| \)
for each \( a \in \text{dom}(g) \) and there is an injection \( k \) of \( \Pi_{a \in \text{dom}(g)} A^g(a) \)
\( \text{into dom(g) \times \kappa} \) such that \( \forall \langle a, s \rangle \in \Pi_{a \in \text{dom}(g)} A^g(a), k(\langle a, s \rangle) = \langle a, \xi \rangle \)
for some \( \xi < \kappa. \)

Let \( f \) be a function of \( \text{dom}(g) \) into \( \mathcal{P}(B) \) such that \( 1 \leq |f(x)| \)
for any \( x \in \text{dom}(g) \) and there is a surjection \( i \) of \( \Pi_{x \in \text{dom}(g)} A^g(x) \)
onto \( \bigcup \text{ran}(f) \) such that for any \( \langle x, s \rangle \in \Pi_{x \in \text{dom}(g)} A^g(x), i(\langle x, s \rangle) \in f(x) \subseteq B. \)

Assume \( B \) is a \( \kappa \)-complete Boolean algebra and for any function \( h \) of \( \text{dom}(f) \times A \) into \( \mathcal{P}(B) \) with \( 1 \leq |h(\langle x, a \rangle)| \leq |g(x)| \)
\( \langle x, a \rangle \in \text{dom}(f) \times A, B \) is \( \langle \lambda, h \rangle \)-distributive. Then, \( B \) is \( \langle \lambda, f \rangle \)-distributive.

**Proof.** Assume that \( 0 < b \leq \bigvee f(x) \) for each \( x \in \text{dom}(f). \) Clearly \( b \leq \bigwedge_{x \in \text{dom}(f)} \bigvee s \in A^g(x) i(\langle x, s \rangle). \) Since \( B \) is \( \kappa \)-complete and \( \{k^{-1}(\langle x, \xi \rangle) : \xi < \kappa \} = A^g(x) \) for each \( x \in \text{dom}(f), \) we may assume that if \( s_1, s_2 \in A^g(x) \) and \( s_1 \neq s_2 \) then \( i(\langle x, s_1 \rangle) \land i(\langle x, s_2 \rangle) = 0. \) That is, there exists a function \( f \) of \( \text{dom}(g) \) into \( \mathcal{P}(B) \) such that \( \text{for each } x \in \text{dom}(g), \text{ \( \hat{f}(x) \) is disjoint, } \bigvee \hat{f}(x) = \bigwedge f(x) \) and for any \( d \in f(x), \) there is a \( d \prime \in f(x) \) with \( d \leq d \prime. \)

Putting \( b(x, a, e) = \bigvee\{i(\langle x, s \rangle) : s \in A^g(x) \text{ and } s(a) = e\} \) for \( \langle x, a \rangle, e \in \Pi_{x \in \text{dom}(f)} A \times g(x), \) we can see that for any \( s \in A^g(x), \)
\[ i(\langle x, s \rangle) = \bigwedge_{a \in A} b(x, a, s(a)) \text{ and so} \]
\[ b \leq \bigwedge_{x \in \text{dom}(f)} \bigvee_{s \in A^g(x)} i(\langle x, s \rangle) \]
\[ = \bigwedge_{x \in \text{dom}(f)} \bigvee_{s \in A} g(x) \bigwedge_{a \in A} b(x, a, s(a)) \]
\[ \leq \bigwedge_{x \in \text{dom}(f)} \bigwedge_{a \in A} \bigvee_{e \in g(x)} b(x, a, e) \]
\[ = \bigwedge_{\langle x, a \rangle \in \text{dom}(f) \times A} \bigvee_{e \in g(x)} b(x, a, e). \]

Define a function \( h \) on \( \text{dom}(f) \times A \) by
\[ h(\langle x, a \rangle) = \{ b \wedge b(x, a, e) : e \in g(x) \} \text{ for } \langle x, a \rangle \in \text{dom}(f) \times A. \]
Then \( 1 \leq |h(\langle x, a \rangle)| \leq |g(x)|. \)

Since \( b = \bigwedge_{\langle x, a \rangle \in \text{dom}(f) \times A} h(\langle x, a \rangle) \) and \( B \) is \( \langle \lambda, h \rangle \)-distributive
by the assumption, there exists a function \( \tilde{v} \in \Pi h \) such that \( \forall t \in [\text{dom}(h)]^{<\lambda} b \wedge \bigwedge_{\langle x, a \rangle \in t} \tilde{v}(x, a) > 0. \)

For each \( x \in \text{dom}(h) \), we define a function \( s_x \) in \( A^g(x) \) by
\[ s_x(a) = y \text{ iff } \tilde{v}(x, a) = b \wedge b(x, a, y) \text{ for } a \in A. \]
Then the function \( v \) on \( \text{dom}(f) \) defined by \( v(x) = i(\langle x, s_x \rangle) \) for \( x \in \text{dom}(f) \) satisfies that \( v \in \Pi f \) and \( \forall t \in [\text{dom}(f)]^{<\lambda} b \wedge \bigwedge_{x \in t} v(x) > 0. \)

For, since \( t \times A \in [\text{dom}(f) \times A]^{<\lambda} = [\text{dom}(h)]^{<\lambda} \), it holds that
\[ b \wedge \bigwedge_{x \in t} v(x) = b \wedge \bigwedge_{x \in t} i(\langle x, s_x \rangle) = b \wedge \bigwedge_{x \in t} b(x, a, s_x(a)) = \bigwedge_{\langle x, a \rangle \in t \times A} (b \wedge b(x, a, s_x(a))) = \bigwedge_{\langle x, a \rangle \in t \times A} \tilde{v}(x, a) > 0. \]

Thus, \( B \) is \( \langle \lambda, f \rangle \)-distributive.

**Corollary 1.** Let \( A, X \) and \( Y \) be non-empty sets such that \( |A| = |A \times X|, |X| < \lambda \) and \( |X^Y| < \kappa. \) Assume that \( B \) is \( \kappa \)-complete. Then each \( \kappa \)-complete \( \langle \lambda, C_{A,Y} \rangle \)-distributive Boolean algebra is \( \langle \lambda, C_{A \times X, Y} \rangle \)-distributive.

**Corollary 2.** Assume that \( \forall \eta \leq \nu \leq \mu, \nu < \lambda \) and \( |\nu| < \kappa. \) Then each \( \kappa \)-complete \( \langle \lambda, (\mu; \nu) \rangle \)-distributive Boolean algebra is \( \langle \lambda, (\mu; |\nu|) \rangle \)-distributive.

**References**


