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Notes on certain non-analytic functions
- Dedicated to the centennial anniversary of the birth of Professor Yusaku Komatu -

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Abstract

Let $\mathcal{N}$ be the class of functions $f(z,\overline{z})$ which are non-analytic in the open unit disk $\mathbb{U}$. Many classes of analytic functions $f(z)$ in $\mathbb{U}$ are studied by mathematicians around the world. There are only few papers for non-analytic functions $f(z,\overline{z})$ in $\mathbb{U}$. The purpose of this paper is to discuss some properties of non-analytic functions $f(z,\overline{z})$ in $\mathbb{U}$ with some examples.

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1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$(1.1)\qquad f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}\,|\,|z|<1\}$. If $f(z)\in\mathcal{A}$ maps the unit circle on to the starlike curve with respect to the origin, then $f(z)$ is said to be starlike with respect to the origin in $\mathbb{U}$. Also, if $f(z)\in\mathcal{A}$ maps the unit circle on to the convex curve, then $f(z)$ is said to be convex in $\mathbb{U}$.

A function $f(z)\in\mathcal{A}$ is said to be starlike of order α if it satisfies

$$(1.2)\qquad \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha\quad(z\in\mathbb{U})$$

for some real α $(0\leqq\alpha<1)$. Also, a function $f(z)\in\mathcal{A}$ is said to be convex of order α if it satisfies

$$(1.3)\qquad \mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha\quad(z\in\mathbb{U})$$

for some real α $(0\leqq\alpha<1)$. It is well known that $f(z)$ is convex of order α in $\mathbb{U}$ if and only if $zf'(z)$ is starlike of order α in $\mathbb{U}$ (see Robertson [3], Komatu [2], Goodman [1]). We see that

$$(1.4)\qquad f(z)=\frac{z}{(1-z)^{2(1-\alpha)}}\quad(z\in\mathbb{U})$$

is starlike of order α in $\mathbb{U}$ and that

$$(1.5)\qquad f(z)=\begin{cases}\dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & \left(\alpha\neq\dfrac{1}{2};z\in\mathbb{U}\right)\\[2ex] -\log(1-z) & \left(\alpha=\dfrac{1}{2};z\in\mathbb{U}\right)\end{cases}$$

is convex of order α in $\mathbb{U}$.

Let $\mathcal{N}$ denote the class of functions

$$(1.6)\qquad f(z,\overline{z})=\sum_{n=1}^{\infty}f_n(z,\overline{z})$$

defined in $\mathbb{U}$, where $z=x+iy$ and $\overline{z}=x-iy$. If $f(z,\overline{z})\in\mathcal{N}$ maps $\mathbb{U}$ onto the starlike domain, then we say that $f(z,\overline{z})$ is non-analytic and starlike in$\mathbb{U}$. Further, we say that $f(z,\overline{z})$ is non-analytic and convex in $\mathbb{U}$ if it maps $\mathbb{U}$ onto the comvex domain. If we consider a function $f(z,\overline{z})$ given by

$$(1.7)\qquad f(z,\overline{z})=\frac{\overline{z}}{(1-\overline{z})^2}=\overline{z}+\sum_{n=2}^{\infty}n\overline{z}^n,$$

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then $f(z,\overline{z})$ is non-analytic and starlike in $\mathbb{U}$, and we see that a function

$$(1.8) \qquad f(z,\overline{z}) = \frac{\overline{z}}{1-\overline{z}} = \overline{z} + \sum_{n=2}^{\infty} \overline{z}^n$$

is non-analytic and convex in $\mathbb{U}$. But if we consider

$$(1.9) \qquad f(z,\overline{z}) = \frac{\overline{z}}{(1-z)^2} = \overline{z} + |z|^2 \sum_{n=2}^{\infty} n z^{n-2},$$

then $f(z,\overline{z})$ is non-analytic, but it is not starlike in $\mathbb{U}$. Also, if we take

$$(1.10) \quad f(z,\overline{z}) = \frac{z}{1-\overline{z}} = z + |z|^2 \sum_{n=2}^{\infty} \overline{z}^{n-2},$$

then $f(z,\overline{z})$ is non-analytic, but it is not convex in $\mathbb{U}$.

2 Some examples

Let us cnsider a function $f(z) \in \mathcal{A}$ defined by

$$(2.1) \qquad f(z) = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 \quad (z \in \mathbb{U}).$$

It follws that

$$(2.2) \qquad \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) = \mathrm{Re}\left(\frac{6(1+z+z^2)}{6+3z+2z^2}\right).$$

If we take $z = e^{i\theta}$ for (2.2), then we have that

$$(2.3) \qquad \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) = \mathrm{Re}\left(\frac{6(2\cos\theta+1)(8\cos\theta+3)}{25+48\cos\theta+48\cos^2\theta}\right).$$

Therefore, for θ satisfying $-\frac{1}{2} \leqq \cos\theta \leqq -\frac{3}{8}$, we see that

$$(2.4) \qquad \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \leqq 0.$$

This means that $f(z)$ is analytic in $\mathbb{U}$, but not starlike in $\mathbb{U}$ by means of (1.2).
Next, we try to consider a function $f(z,\overline{z}) \in \mathcal{N}$ given by

$$(2.5) \quad f(z,\overline{z}) = z + \frac{1}{2}z\overline{z} + \frac{1}{3}z\overline{z}^2 \quad (z \in \mathbb{U}).$$

Taking $z = re^{i\theta}\,(0 \leqq r < 1; 0 \leqq \theta < 2\pi)$, we have that

$$(2.6) \qquad f(z,\overline{z}) = \left(\frac{1}{2}r^2 + \left(r + \frac{1}{3}r^3\right)\cos\theta\right) + i\left(r - \frac{1}{3}r^3\right)\sin\theta.$$

If we write that

$$(2.7) \quad u = \mathrm{Re} f(z,\overline{z}) = \frac{1}{2}r^2 + \left(r + \frac{1}{3}r^3\right)\cos\theta$$

and

$$(2.8) \qquad v = \mathrm{Im} f(z,\overline{z}) = \left(r - \frac{1}{3}r^3\right)\sin\theta,$$

then it follows that

$$(2.9) \qquad \frac{\left(u - \frac{1}{2}r^2\right)^2}{\left(r + \frac{1}{3}r^3\right)^2} + \frac{v^2}{\left(r - \frac{1}{3}r^3\right)^2} = 1$$

for $0 \leqq r < 1$.
Letting $r \to 1$, we obtain that

$$(2.10) \qquad \frac{\left(u - \frac{1}{2}\right)^2}{\left(\frac{4}{3}\right)^2} + \frac{v^2}{\left(\frac{2}{3}\right)^2} = 1$$

which shows the elliptic domain including the origin. Thus, the function $f(z,\overline{z})$ maps $\mathbb{U}$ onto the outside of the elliptic domain which is convex in $\mathbb{U}$.
Furthermore, if we define $f(z,\overline{z})$ by

$$(2.11) \qquad f(z,\overline{z}) = \overline{z} + \frac{1}{2}z\overline{z} + \frac{1}{3}z^2\overline{z} \quad (z \in \mathbb{U}),$$

then

$$(2.12)\quad f(z,\overline{z})=\left(\frac{1}{2}r^2+\left(r+\frac{1}{3}r^3\right)\cos\theta\right)+i\left(\frac{1}{3}r^3-r\right)\sin\theta$$

with $z=re^{i\theta}\,(0\leqq r<1, 0\leqq\theta<2\pi)$. Thus, writing that

$$(2.13)\quad u=\mathrm{Re}f(z,\overline{z})=\frac{1}{2}r^2+\left(r+\frac{1}{3}r^3\right)\cos\theta$$

and

$$(2.14)\quad v=\mathrm{Im}f(z,\overline{z})=\left(\frac{1}{3}r^3-r\right)\sin\theta,$$

we have that

$$(2.15)\quad \frac{\left(u-\frac{1}{2}r^2\right)^2}{\left(r+\frac{1}{3}r^3\right)^2}+\frac{v^2}{\left(\frac{1}{3}r^3-r\right)^2}=1$$

for $0\leqq r<1$. Therefore, making $r\to 1$ in (2.15), we have the following elliptic equation

$$(2.16)\quad \frac{\left(u-\frac{1}{2}\right)^2}{\left(\frac{4}{3}\right)^2}+\frac{v^2}{\left(\frac{2}{3}\right)^2}=1.$$

Consequently, $f(z,\overline{z})$ maps $\mathbb{U}$ onto the outside of the elliptic domain which is convex in $\mathbb{U}$. Finally, we see that $f(z,\overline{z})$ in (2.1) is the sense preserving mapping and that $f(z,\overline{z})$ in (2.11) is the sense reversing mapping.

3 Some properties of non-analytic functions

In order to discuss our problems for non-analytic functions, we have to recall here the following lemma due to Silverman [4].

Lemma 1 *If an analytic function*

$$(3.1)\quad f(z)=\sum_{n=1}^{\infty}a_nz^n\quad (a_1\neq 0; z\in\mathbb{U})$$

satisfies

$$(3.2)\quad \sum_{n=2}^{\infty}n^2|a_n|\leqq|a_1|,$$

then $f(z)$ is convex in $\mathbb{U}$.

Lemma 2 *If a function $f(z)$ given by (3.1) is analytic and convex in $\mathbb{U}$, then for all θ $(0<\theta<2\pi)$, $r|f'(re^{i\theta})|$ is increasing for r $(z=re^{i\theta}, r\neq 0)$, that is, that*

$$(3.3)\quad r_1|f'(r_1e^{i\theta})|<r_2|f'(r_2e^{i\theta})|$$

for $0<r1<r_2<1$.

Proof For a fixed θ $(0\leqq\theta<2\pi)$, we consider $\phi(r)$ by

$$(3.4)\quad \phi(r)=\log|rf'(re^{i\theta})|\quad (0<r<1).$$

Then, we have that

$$(3.5)\quad \begin{aligned}\frac{d}{dr}\phi(r)&=\frac{\partial}{\partial r}\left(\log r+\log|f'(re^{i\theta})|\right)\\&=\frac{1}{r}+\frac{\partial}{\partial r}\left\{\mathrm{Re}\left(\log f'(re^{i\theta})\right)\right\}\\&=\frac{1}{r}+\mathrm{Re}\left(\frac{f''(re^{i\theta})}{f'(re^{i\theta})}e^{i\theta}\right)\\&=\frac{1}{r}\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0,\end{aligned}$$

because $f(z)$ is convex in $\mathbb{U}$. This gives us that $r|f'(re^{i\theta})|$ is increasing for r $(0<r<1)$.

With the help of Lemma 1 and Lemma 2, we derive

Theorem 1 *Let $f(z,\overline{z})$ given by (1.5) be non-analytic in $\mathbb{U}$. Also let*

$$(3.6)\quad f_n(rz,r\overline{z})=r^nf_n(z,\overline{z})\quad (0<r<1).$$

If $f(z,\overline{z})$ satisfies

$$(3.7)\quad \sum_{n=2}^{\infty} n|f_n(e^{i\theta}, e^{-i\theta})| \leqq |f_1(e^{i\theta}, e^{-i\theta})| \neq 0$$

for all θ $(0 \leqq \theta < 2\pi)$*, then* $f(z,\overline{z})$ *is increasing for* r $(0 < r < 1)$*, that is*

$$(3.8)\qquad |f(z_1, \overline{z_1})| < |f(z_2, \overline{z_2})|$$

for $z_1 = r_1 e^{i\theta}$ *and* $z_2 = r_2 e^{i\theta}$ *with* $0 < r_1 < r_2 < 1$.

Corollary 1 *Let* $f(z,\overline{z})$ *given by* (1.6) *satisfy the conditions* (3.6) *and* (3.7). *Then* $|f(z,\overline{z})|$ *is radially increasing in* $\mathbb{U}$.

Proof Let us consider the function $F_z(u)$ defined by

$$(3.9)\quad F_z(u) = \sum_{n=1}^{\infty} \left(\frac{f_n(z,\overline{z})}{nz^n} \right) u^n \quad (u \in \mathbb{U})$$

for $f(z,\overline{z})$. Then, $F_z(u)$ has a radius R such that

$$(3.10)\qquad R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{\left| \dfrac{f_n(z,\overline{z})}{nz^n} \right|}}$$

$$= \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{\left| \dfrac{f_n(e^{i\theta}, e^{-i\theta})|z|^n}{nz^n} \right|}}$$

$$= \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{\left| \dfrac{f_n(e^{i\theta}, e^{-i\theta})}{n} \right|}}$$

$$\geqq \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{\left| \dfrac{f_1(e^{i\theta}, e^{-i\theta})}{n^2} \right|}} = \frac{1}{M} = 1,$$

where

$$(3.11)\qquad M = \limsup_{n\to\infty} \frac{\frac{1}{(n+1)^2}|f_1(e^{i\theta}, e^{-i\theta})|}{\frac{1}{n^2}|f_1(e^{i\theta}, e^{-i\theta})|}.$$

Thus, $F_z(u)$ is analytic in $\mathbb{U}$. Next, applying (3.7), we know that

$$(3.12)\qquad \sum_{n=2}^{\infty} n^2 \left| \frac{f_n(z,\overline{z})}{nz^n} \right|$$

$$= \sum_{n=2}^{\infty} n \left| f_n\left(\frac{z}{|z|}, \frac{\overline{z}}{|z|} \right) \right|$$

$$= \sum_{n=2}^{\infty} n |f_n(e^{i\theta}, e^{-i\theta})|$$

$$\leqq |f_1(e^{i\theta}, e^{-i\theta})| = \left| \frac{f_1(z,\overline{z})}{z} \right|,$$

which implies that $F_z(u)$ is convex in $\mathbb{U}$ by Lemma 1.
Furthermore, by using Lemma 2, we have that

$$(3.13)\qquad |u_1 F_z'(u_1)| < |u_2 F_z'(u_2)|$$

for $u_1 = \rho_1 e^{i\phi}, u_2 = \rho_2 e^{i\phi}$, and $0 < \rho_1 < \rho_2 < 1$. If we take $u_1 = z$ and $u_2 = rz$ with $1 < r < \dfrac{1}{|z|}$, (3.13) becomes that

$$(3.14)\qquad |zF_z'(z)| < |rzF_z'(rz)|.$$

It follows that

$$(3.15)\qquad \left| z \sum_{n=1}^{\infty} \left(\frac{f_n(z,\overline{z})}{nz^n} \right) nz^{n-1} \right|$$

$$< \left| rz \sum_{n=1}^{\infty} \left(\frac{f_n(z,\overline{z})}{nz^n} nr^{n-1} z^{n-1} \right) \right|$$

for $1 < r < \dfrac{1}{|z|}$. This shows us that

$$(3.16)\qquad \left| \sum_{n=1}^{\infty} f_n(z,\overline{z}) \right| < \left| \sum_{n=1}^{\infty} f_n(rz, r\overline{z}) \right|,$$

that is, that

$$(3.17)\qquad |f(z,\overline{z})| < |f(rz, r\overline{z})|$$

for $1 < r < \dfrac{1}{|z|}$. Therefore, we conclude that $|f(z,\overline{z})|$ is increasing for r.

We also have that

Corollary 2 *If a function*

$$(3.18)\qquad f(z,\overline{z})=\sum_{n=1}^{\infty}a_n z^{k_n}\overline{z}^{n-k_n}\quad (a_1\neq 0)$$

is non-analytic in $\mathbb{U}$ *and satisfies*

$$(3.19)\qquad \sum_{n=2}^{\infty}n|a_n|\leqq |a_1|,$$

then $|f(z,\overline{z})|$ *is increasing for* $|z|$.

Finally, we consider one example for Corollary 2.

Example 1 Let us consider

$$(3.20)\qquad f(z,\overline{z})=z+\frac{1}{4}z\overline{z}+\frac{1}{9}z^2\overline{z}+\frac{1}{24}z^3\overline{z}$$

which is non-analytic in $\mathbb{U}$. If we take $z=re^{i\theta}$, then (3.20) becomes

$$(3.21)\qquad f(z,\overline{z}) = re^{i\theta}+\frac{1}{4}r^2+\frac{1}{9}r^3e^{i\theta}+\frac{1}{24}r^4e^{i2\theta}.$$

This gives us that

$$(3.22)\qquad \sum_{n=2}^{\infty}n|a_n|=1.$$

Further, we see that

$$(3.23)\qquad |f(z,\overline{z})|^2 = \left\{\frac{1}{4}r^2+\left(1+\frac{1}{9}r^2\right)r\cos\theta+\frac{1}{24}r^4\cos 2\theta\right\}^2 + \left\{\left(1+\frac{1}{9}r^2\right)r\sin\theta+\frac{1}{24}r^4\sin 2\theta\right\}^2.$$

Finally, using the computer, we know that $|f(z,\overline{z})|^2$ is increasing for r $(0<r<1)$.
For example, if we take $\theta=0$, then we have that

$$(3.24)\qquad |f(z,\overline{z})|^2 = r^2\left(1+\frac{1}{4}r+\frac{1}{9}r^2+\frac{1}{24}r^3\right)^2$$

and see that $|f(z,\overline{z})|^2$ is increasing for r $(0<r<1)$.
If we consider $\theta=\frac{\pi}{2}$, then we obtain

$$(3.25)\qquad |f(z,\overline{z})|^2=\frac{1}{4}r^3\left(1-\frac{1}{6}r^2\right)^2.$$

Thus we see that $|f(z,\overline{z})|^2$ in (3.25) is increasing for r $(0<r<1)$.

Furthermore, if we let $\theta=\pi$, then $|f(z,\overline{z})|^2$ becomes that

$$(3.26)\qquad |f(z,\overline{z})|^2 = r^2\left(-1+\frac{1}{4}r-\frac{1}{9}r^2+\frac{1}{24}r^3\right)^2.$$

Also we conclude that $|f(z,\overline{z})|^2$ is increasing for r $(0<r<1)$.

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