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Image domains of certain starlike functions

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Abstract

Let $\mathcal{S}^*$ be the class of analytic functions $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ which are starlike with respect to the origin in the open unit disk $\mathbb{U}$. We discuss the length of the image curve of $f(z)$ and the area of the image domain of $f(z)$ in the present paper.

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1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be univalent in $\mathbb{U}$ if and only if $f(z_1) \neq f(z_2)$ for $z_1 \in \mathbb{U}$ and $z_2 \in \mathbb{U}$ such that $z_1 \neq z_2$. The class of all univalent functions $f(z)$ in $\mathbb{U}$ is denoted by $\mathcal{S}$. If $f(z) \in \mathcal{A}$ satisfies the condition given in

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}), \tag{1.2}$$

then $f(z)$ is called starlike with respect to the origin in $\mathbb{U}$. Also, we denote by $\mathcal{S}^*$ all starlike functions $f(z)$ with respect to the origin in $\mathbb{U}$. Moreover, if $f(z) \in \mathcal{A}$ satisfies $zf'(z) \in \mathcal{S}^*$ which is equivalent to

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$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \qquad (z\in\mathbb{U}), \tag{1.3}$$

then we claim that $f(z)$ is convex in $\mathbb{U}$ and we will formalize it as $f(z)\in\mathcal{K}$ (cf. Duren [1]).

Then, it is well-known that

$$f(z)=\frac{z}{(1-z)^2}=z+\sum_{n=2}^{\infty}nz^n \tag{1.4}$$

is the extremal function for $\mathcal{S}^*$ and that

$$f(z)=\frac{z}{1-z}=z+\sum_{n=2}^{\infty}z^n \tag{1.5}$$

is the extremal function for $\mathcal{K}$ (cf. Robertson [2]).

In 1972, Silverman [3] showed that if $f(z)\in\mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty}n|a_n|\leqq 1, \tag{1.6}$$

then we will get $f(z)\in\mathcal{S}^*$, and that if $f(z)\in\mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty}n^2|a_n|\leqq 1, \tag{1.7}$$

then $f(z)\in\mathcal{K}$. With the results obtained by Silverman [3], it is already known that a function $f(z)$ given by

$$f(z)=z+\frac{1}{n}z^n \qquad (n=2,3,4,\cdots) \tag{1.8}$$

is in the class $\mathcal{S}^*$ and that a function

$$f(z)=z+\frac{1}{n^2}z^n \qquad (n=2,3,4,\cdots) \tag{1.9}$$

is in the class $\mathcal{K}$.

From now on, we are going to consider the image domains of $f(z)$ given by (1.8) for $z\in\mathbb{U}$.

2 Limacon

Now, at first let us consider the function

$$f(z) = z + \frac{1}{n} z^n \qquad (n = 2, 3, 4, \cdots) \tag{2.1}$$

which is called the limacon in $\mathbb{U}$. If we write that $z = re^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$ and $f(z) = u + iv$ in (2.1), then we know that

$$u = r\cos\theta + \frac{r^n}{n}\cos n\theta \tag{2.2}$$

and

$$v = r\sin\theta + \frac{r^n}{n}\sin n\theta. \tag{2.3}$$

We confirm that $f(z) \in \mathcal{S}^*$.

Let us suppose that $\mathcal{L}_r$ denotes the length of the image curve of $f(z)$ for $|z| = r$, and that $\mathcal{S}_r$ is the area of the image domain of $f(z)$ for $|z| < r$.

Theorem 2.1 *If $f(z)$ is given by* (2.1), *then we have*

$$\mathcal{L}_r = 2r(n-1)\int_0^{\frac{\pi}{n-1}} \sqrt{1 + r^{2(n-1)} + 2r^{n-1}\cos(n-1)\theta}\; d\theta \tag{2.4}$$

for $r > 0$, and we also have

$$\mathcal{S}_r = \frac{r^2\left(n + r^{2(n-1)}\right)}{n}\pi \tag{2.5}$$

for $0 < r \leqq 1$.

Proof. By means of the definition for $\mathcal{L}_r$, we have

$$\begin{aligned} \mathcal{L}_r &= \int_0^{2\pi} \sqrt{\left(\frac{\partial u}{\partial\theta}\right)^2 + \left(\frac{\partial v}{\partial\theta}\right)^2}\; d\theta \\ &= \int_0^{2\pi} \sqrt{\left(r\sin\theta + r^n \sin n\theta\right)^2 + \left(r\cos\theta + r^n\cos n\theta\right)^2}\; d\theta \\ &= 2r(n-1)\int_0^{\frac{\pi}{n-1}} \sqrt{1 + r^{2(n-1)} + 2r^{n-1}\cos(n-1)\theta}\; d\theta. \end{aligned} \tag{2.6}$$

Furthermore, we will get

$$\mathcal{S}_r = \int_\pi^0 v\, du - \int_\pi^{2\pi} v\, du \tag{2.7}$$

$$= \int_0^{2\pi} \left(r^2 \sin^2\theta + \frac{n+1}{n} r^{n+1} \sin\theta \sin n\theta + \frac{1}{n} r^{2n} \sin^2 n\theta \right) d\theta$$

$$= \frac{r^2\left(n + r^{2(n-1)}\right)}{n}\pi.$$

The deduction provided in (2.6) and (2.7) leads to the validity of the theorem. □

If we consider the case of $r = 1$ in Theorem 2.1, then we will get what follows.

Corollary 2.1 *If $f(z)$ is given by (2.1), then $\mathcal{L}_1 = 8$ and $\mathcal{S}_1 = \frac{n+1}{n}\pi$.*

Remark 2.1 Corollary 2.1 shows us that $\mathcal{L}_1 = 8$ and $\mathcal{S}_1 = \frac{n+1}{n}\pi$ for any n ($n = 2, 3, 4, \cdots$) if $r = 1$. Furthermore, $\lim_{n\to\infty} \mathcal{S}_r = \pi r^2$.

Corollary 2.2 *If $f(z)$ is indicated in (2.1), then (2.8) naturally follows.*

$$\mathcal{L}_r = r^{n+1}\mathcal{L}_{\frac{1}{r}} \qquad (r > 0). \tag{2.8}$$

Proof. In view of $\mathcal{L}_r$ in (2.4), we calculate $\mathcal{L}_{\frac{1}{r}}$ as follows

$$\mathcal{L}_{\frac{1}{r}} = \frac{2(n-1)}{r} \int_0^{\frac{\pi}{n-1}} \sqrt{1 + \left(\frac{1}{r}\right)^{2(n-1)} + 2\left(\frac{1}{r}\right)^{n-1} \cos(n-1)\theta}\ d\theta \tag{2.9}$$

$$= \frac{2(n-1)}{r^n} \int_0^{\frac{\pi}{n-1}} \sqrt{1 + r^{2(n-1)} + 2r^{n-1}\cos(n-1)\theta}\ d\theta = \frac{1}{r^{n+1}}\mathcal{L}_r$$

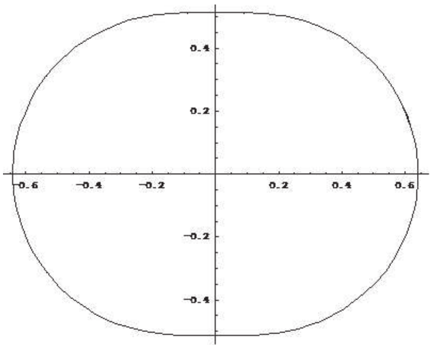
for $n = 2, 3, 4, \cdots$ and $r > 0$. □

3 Case of $n = 3$

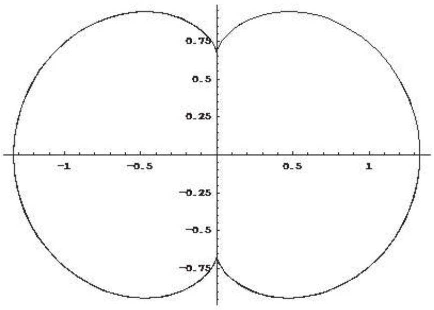
In this section, we now analyze the case of $n = 3$ in detail. We have to check that the function

$$f(z) = z + \frac{1}{3}x^3 \qquad (0 < r \leqq \sqrt{3}) \tag{3.1}$$

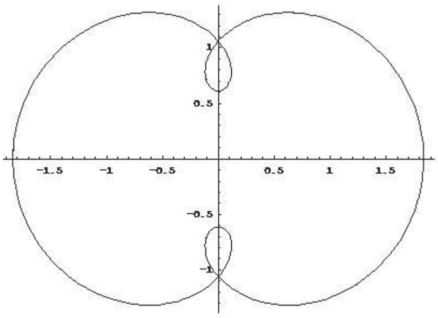
maps $|z| = r$ for the following curves.



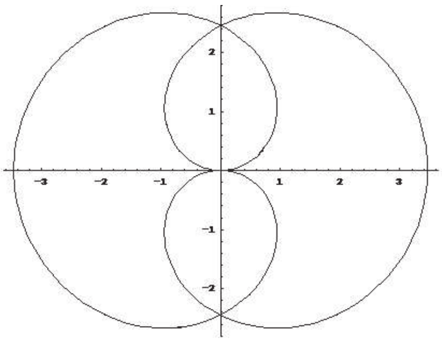
$0 < r < 1$



$r = 1$



$1 < r < \sqrt{3}$



$r = \sqrt{3}$

From the four figures shown above, we can now derive

Theorem 3.1 *If $f(z)$ is given by (3.1) with $0 < r \leqq \sqrt{3}$, then we have*

$$2r(1-r^2)\pi \leqq \mathcal{L}_r < 8 \qquad (0 < r < 1), \tag{3.2}$$

$$\mathcal{L}_r = 8 \qquad (r = 1), \tag{3.3}$$

$$8 < \mathcal{L}_r \leqq 2r(1+r^2)\pi \qquad (1 < r < \sqrt{3}) \tag{3.4}$$

and

$$4\sqrt{3}\,\pi \leqq \mathcal{L}_r \leqq 8\sqrt{3}\,\pi \qquad (r = \sqrt{3}). \tag{3.5}$$

Proof. We can claim that $z = re^{i\theta}$ and $f(z) = u + iv$ for $f(z)$ of (3.1). Then

$$\begin{cases} u = r\cos\theta + \dfrac{r^3}{3}\cos 3\theta \\[2ex] v = r\sin\theta + \dfrac{r^3}{3}\sin 3\theta \end{cases} \tag{3.6}$$

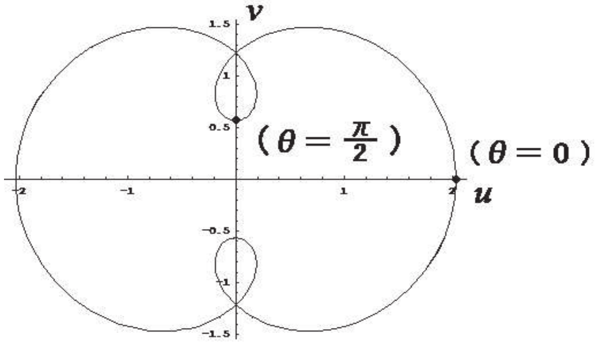
and

$$(3.7)\qquad \begin{cases} \dfrac{\partial u}{\partial \theta} = -r\sin\theta - r^3 \sin 3\theta \\[2ex] \dfrac{\partial v}{\partial \theta} = r\cos\theta + r^3\cos 3\theta. \end{cases}$$

If $0 < r < 1$, then

$$(3.8)\qquad \begin{aligned} \mathcal{L}_r &= \int_0^{2\pi} \sqrt{\left(\frac{\partial u}{\partial \theta}\right)^2 + \left(\frac{\partial v}{\partial \theta}\right)^2}\, d\theta \\ &= 4r\int_0^{\frac{\pi}{2}} \sqrt{1 + r^4 + 2r^2\cos 2\theta}\, d\theta. \end{aligned}$$

This provide us with (3.2) for $0 < r < 1$. If $r = 1$, then we obtain that $\mathcal{L}_r = 8$. If $1 < r < \sqrt{3}$, then we have the following image domain by $f(z)$ for $1 < |z| < \sqrt{3}$.



Using (3.8), we can confirm that

$$(3.9)\qquad 8 < \mathcal{L}_r \leqq 4r\int_0^{\frac{\pi}{2}} (1 + r^2)\, d\theta = 2r(1 + r^2)\pi$$

for $1 < r < \sqrt{3}$. Finally, if $r = \sqrt{3}$, then (3.8) becomes

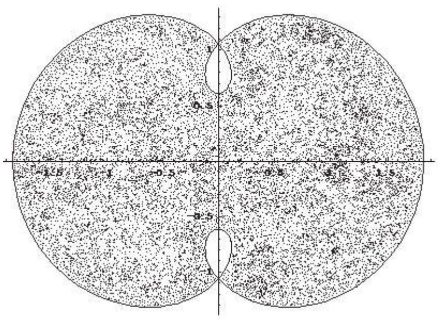
$$(3.10)\qquad \mathcal{L}_r = 4\sqrt{6}\int_0^{\frac{\pi}{2}} \sqrt{5 + 3\cos 2\theta}\, d\theta.$$

Therefore, we obtain the following inequality

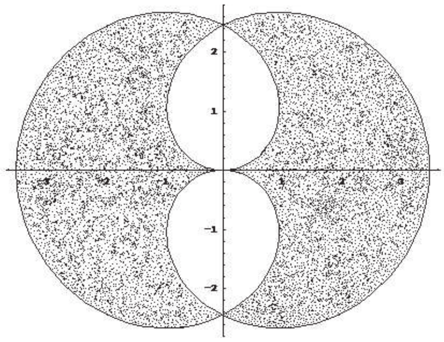
$$(3.11)\qquad 8\sqrt{3}\int_0^{\frac{\pi}{2}} d\theta \leqq \mathcal{L}_r \leqq 16\sqrt{3}\int_0^{\frac{\pi}{2}} d\theta$$

which gives (3.5) for $r = \sqrt{3}$. □

Next, we will consider the area of the image of $f(z)$ for $|z| = r$. If $0 < r \leqq 1$ is valid, then the image of $f(z)$ is the starlike domain. Thus, we use $\mathcal{S}_r$ for the area of the image for $0 < r \leqq 1$. In the case of $1 < r \leqq \sqrt{3}$, $f(z)$ is not starlike as in the following figures.



$1 \leqq r \leqq \sqrt{3}$



$r = \sqrt{3}$

In this case, we regard the shaded parts as $\mathcal{S}_r$.

Theorem 3.2 *If $f(z)$ is given by* (3.1) *with $0 < r \leqq \sqrt{3}$, then we have*

$$\mathcal{S}_r = \frac{r^2(3+r^4)}{3}\pi \qquad (0 < r < 1), \tag{3.12}$$

$$\mathcal{S}_r = \frac{4}{3}\pi \qquad (r = 1), \tag{3.13}$$

$$\mathcal{S}_r = r^2\left(1+\frac{r^4}{3}\right)\left(4\cos^{-1}\left(\frac{\sqrt{3(r^2-1)}}{2r}\right)-\pi\right) + \frac{4r^2}{3}\sqrt{3(r^2-1)(r^2+3)} \qquad (1 < r < \sqrt{3}) \tag{3.14}$$

and

$$\mathcal{S}_r = 24 \qquad (r = \sqrt{3}). \tag{3.15}$$

Proof. It is clear that $\mathcal{S}_r$ satisfies (3.12) for $0 < r < 1$ and (3.13) for $r = 1$ from Theorem 2.1. Thus, we only need to argue for $1 < r \leqq \sqrt{3}$. We consider that

$$u = r\cos\theta + \frac{r^3}{3}\cos 3\theta = r\cos\theta\left(1 - r^2 + \frac{4r^2}{3}\cos^2\theta\right) = 0 \tag{3.16}$$

for $0 \leqq \theta \leqq \dfrac{\pi}{2}$. It follows that

$$\theta = \frac{\pi}{2} \quad \text{and} \quad \theta = \cos^{-1}\left(\frac{\sqrt{3(r^2-1)}}{2r}\right).$$

Letting $\theta_1 = \dfrac{\pi}{2}$ and

$$\theta_2 = \cos^{-1}\left(\frac{\sqrt{3(r^2-1)}}{2r}\right), \tag{3.17}$$

from the discussion above, we can calculate as follows.

$$\begin{aligned}
\mathcal{S}_r &= 4\left\{\int_{\theta_2}^{0} v\,du - \int_{\theta_1}^{\theta_2} v\,du\right\} \qquad (3.18)\\
&= 4\left\{\int_{\theta_2}^{0}\left(r\sin\theta + \frac{r^3}{3}\sin 3\theta\right)\left(-r\sin\theta - r^3\sin 3\theta\right)d\theta\right.\\
&\qquad \left. - \int_{\frac{\pi}{2}}^{\theta_2}\left(r\sin\theta + \frac{r^3}{3}\sin 3\theta\right)\left(-r\sin\theta - r^3\sin 3\theta\right)d\theta\right\}\\
&= \Big[F(\theta)\Big]_{\theta_2}^{0} - \Big[F(\theta)\Big]_{\frac{\pi}{2}}^{\theta_2}\\
&= F(0) - 2F(\theta_2) + F\left(\frac{\pi}{2}\right),
\end{aligned}$$

where

$$F(\theta) = -\frac{2r^2}{3}\left\{(3+r^4)\theta + \frac{4r^2-3}{2}r\sin 2\theta - r^2\sin 4\theta - \frac{r^4}{6}\sin 6\theta\right\}. \tag{3.19}$$

Therefore, using the following formulas

$$\sin\theta_2 = \frac{\sqrt{r^2+3}}{2r}, \tag{3.20}$$

$$\sin 2\theta_2 = \frac{\sqrt{3(r^2-1)(r^2+3)}}{2r^2}, \tag{3.21}$$

$$\sin 4\theta_2 = \frac{(r^2-3)\sqrt{3(r^2-1)(r^2+3)}}{2r^4} \tag{3.22}$$

and

$$\sin 6\theta_2 = \frac{3(3-2r^2)\sqrt{3(r^2-1)(r^2+3)}}{2r^6}, \tag{3.23}$$

we obtain

$$\mathcal{S}_r = \frac{r^2}{3}\left\{(3+r^4)(4\theta_2-\pi) + 2(4r^2-3)\sin 2\theta_2 - 4r^2\sin 4\theta_2 - \frac{2r^4}{3}\sin 6\theta_2\right\} \tag{3.24}$$

$$= r^2 \left(1 + \frac{r^4}{3}\right) \left(4\cos^{-1}\left(\frac{\sqrt{3(r^2-1)}}{2r}\right) - \pi\right) + \frac{4r^2}{3}\sqrt{3(r^2-1)(r^2+3)}$$

for $1 < r < \sqrt{3}$. Finally, letting $r = \sqrt{3}$ in (3.24), we have that

$$\mathcal{S}_r = \frac{4r^2}{3}\sqrt{3(r^2-1)(r^2+3)} = 24, \tag{3.25}$$

because

$$\cos^{-1}\left(\frac{\sqrt{3(r^2-1)}}{2r}\right) = \frac{\pi}{4} \tag{3.26}$$

is valid for $r = \sqrt{3}$. □

Remark For the special r in Theorem 3.2, we have $\mathcal{S}_r = \frac{13}{24}\pi = 1.70169\cdots$ for $r = \frac{1}{\sqrt{2}} < 1$ and $\mathcal{S}_r = \frac{7}{8}\pi + 3\sqrt{3} = 7.94504\cdots$ for $r = \frac{\sqrt{6}}{2} > 1$.

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References

[1] P. L. Duren, *Univalent Functions*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983

[2] M. S. Robertson, *On the theory of univalent functions*, Ann. of Math. **37** (1936), 374–408

[3] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1972), 109–116